Announcements

1) Math Career Talks

Monday 1/28

2:30 CB 1030

2) Supplement to HW 2

due Monday

Span of vectors in $\mathbb{R}^3$

The span of any set of nonzero vectors is either a line in $\mathbb{R}^3$, a plane in $\mathbb{R}^3$, or all of $\mathbb{R}^3$. How do you tell which it is?

Less than three vectors is never $\mathbb{R}^3$ in span

Either a line or a plane.

You get a line if you have either one vector or two vectors with one a scalar multiple of the other.

3 or more vectors in your set

The span is

1) a line if all vectors in the set are scalar multiples of eachother.

2) all of $\mathbb{R}^3$ if you can take three vectors out of your set, say v_1, v_2, and v_3, make the matrix $A = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$, and $\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

3) a plane if neither 1) nor 2).

Example 1: Find the span of

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 18 \end{bmatrix}, \begin{bmatrix} 1 \\ -11 \\ 23 \end{bmatrix} \right\}.$$

The span is not a line.

Is it $\mathbb{R}^3$?

Let $A = \begin{bmatrix} 1 & 6 & 1 \\ 3 & 4 & -11 \\ -1 & 18 & 23 \end{bmatrix}$

The rref of A is

$$\begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so the span is not $\mathbb{R}^3$.

Therefore, it must be a plane.

Example 2': Find the span of

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \\ 18 \end{bmatrix}, \begin{bmatrix} 1 \\ -11 \\ 23 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Not a line.

We know if $A = \begin{bmatrix} 1 & 3 & 1 \\ 3 & 4 & -11 \\ -1 & 18 & 23 \end{bmatrix}$,

rref of A is not $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Not good enough —

Make another matrix choosing three different vectors

$$B = \begin{bmatrix} 1 & 6 & 5 \\ 3 & 4 & 0 \\ -1 & 18 & 1 \end{bmatrix}$$

So rref of B is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and}$$

the span is $\mathbb{R}^3$.

More on vectors

Dot (inner) product

Let $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$

be vectors in $\mathbb{R}^n$. The

dot product of v with w

is the number

$$\boxed{\begin{aligned} v \cdot w &= v_1 w_1 + v_2 w_2 + \cdots + v_n w_n \\ &= \sum_{i=1}^{n} v_i w_i \end{aligned}}$$

Example 3:

Let $v = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $w = \begin{bmatrix} -3 \\ 16 \end{bmatrix}$

in $\mathbb{R}^2$. Then

$$v \cdot w = 1 \cdot (-3) + 4 \cdot (16)$$
$$= \boxed{61}$$

Let $v = \begin{bmatrix} -3 \\ 22 \\ 6 \end{bmatrix}$, $w = \begin{bmatrix} 10 \\ 15 \\ -1 \end{bmatrix}$

in $\mathbb{R}^3$. Then

$$v \cdot w = (-3)\cdot(10) + (22)\cdot(15) + 6\cdot(-1)$$

$$= -30 + 330 - 6$$

$$= \boxed{294}$$

Ax revisited

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and

A be the $m \times n$ matrix

$A = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ where

$v_1, v_2, \ldots, v_m$ are n-vectors

written as rows.

Then Ax

$$= \begin{bmatrix} v_1 \cdot x \\ v_2 \cdot x \\ v_3 \cdot x \\ \vdots \\ v_m \cdot x \end{bmatrix},$$

an m-vector.

Example 4:

$$A = \begin{bmatrix} 1 & -3 \\ 5 & 6 \\ 0 & 30 \end{bmatrix}, \quad x = \begin{bmatrix} -8 \\ 4 \end{bmatrix}$$

2×1

3×2

$$Ax = \begin{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 5 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 4 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 30 \end{bmatrix} \cdot \begin{bmatrix} -8 \\ 4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -20 \\ -16 \\ 120 \end{bmatrix}$$

Norms

Definition: A norm on $\mathbb{R}^n$ is a function $\|\cdot\|$ from $\mathbb{R}^n$ to $[0,\infty)$ satisfying

1) $\|\vec{0}\| = 0$ and $\|v\| = 0$ only when $v = \vec{0}$.

2) If c is a real number and v is a vector,

$$\|cv\| = |c| \cdot \|v\|$$

3) If v and w are vectors,

$$\|v+w\| \leq \|v\| + \|w\|$$

(triangle inequality)

Example 5: Let p be any real number bigger than or equal to one. Define $\|\cdot\|_p$ from $\mathbb{R}^2$ to $[0,\infty)$ by

$$\boxed{\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\| = \left(|x|^p + |y|^p \right)^{1/p}}$$

Special values for p

p=2 $\quad \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2 = \sqrt{x^2+y^2}$

p=1 $\quad \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_1 = |x|+|y|$

p=∞? $\quad \left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_\infty$

$$= \max\{|x|, |y|\}.$$

Angle Between Vectors

If v and w are two vectors in $\mathbb{R}^n$ and v is not a scalar multiple of w, then if both vectors are nonzero, their span is a plane. On that plane, we can find the angle θ between v and w by the formula

$$\boxed{\cos\theta = \frac{v \cdot w}{\|v\|_2 \|w\|_2}}$$

Example 6: Find the

angle between

$$v = \begin{bmatrix} -1 \\ 5 \\ 6 \end{bmatrix} \text{ and } w = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

$$v \cdot w = -4 - 42 = -46$$

$$\|v\|_2 = \sqrt{(-1)^2 + 5^2 + 6^2}$$

$$= \sqrt{62}$$

$$\|w\|_2 = \sqrt{4^2 + (-7)^2}$$

$$= \sqrt{65}$$

Then

$$\cos\theta = \frac{-46}{\sqrt{65}\cdot\sqrt{62}}, \text{ so}$$

$$\theta = \arccos\left(\frac{-46}{\sqrt{65}\cdot\sqrt{62}}\right)$$

Use a calculator or computer to get this value